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Evaluating one-loop integrals at finite temperature

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Abstract. Generic one-loop integrals at finite temperature, appearing in integrations over non-static modes, are given a definite expression in terms of series in powers of $\sim (m/T)^2$, where m is a mass parameter and T is the temperature.

In the imaginary time formalism of field theory at finite temperature [1] it is sometimes useful to express a Feynman integral representing a given Feynman diagram including bosonic propagators as a combination of diagrams whose propagators involve either zero or non-zero Matsubara frequencies, $\omega_n = 2\pi nT$, $n = 0, \pm 1, \pm 2, \ldots$ (or a combination of both in the case of higher-loop diagrams) [2].

We present a short and direct way to evaluate generic one-loop integrals appearing in the evaluation of (amputated) Feynman diagrams including only non-static propagators, i.e., those with non-zero Matsubara frequencies. Using dimensional regularization to regularize those integrals in dimension $D = 3 - 2\varepsilon$, $\varepsilon > 0$, we have that they can always be written as

$$J(m; a, b) = T\mu^{2\varepsilon} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{(k^2)^a}{[k^2 + \omega_n^2 + m^2]^b} \tag{1}$$

where μ is the scale of dimensional regularization, a is an integer greater than or equal to zero and b is a positive integer. When the Feynman integrals depend explicitly on the sum of the external three momenta at a vertex, $\boldsymbol{P}$, integrals of the form (1) can be seen as the coefficients of a Taylor expansion of the Feynman integral around $P^2 = 0$. In the case of a Feynman integral involving denominators of the form

$$[k^2 + \omega_n^2 + m_1^2][(k + P)^2 + \omega_n^2 + m_2^2]$$

the use of Feynman parametrization is understood in order to obtain the form given in equation (1), after Taylor expansion. This means that the mass parameter m in equation (1) will be a function of the variable of integration, say x. The integral over x as well as the function of x appearing there are not written explicitly.

Using the definition of Matsubara frequencies, $\omega_n = 2\pi nT$, and rescaling the momenta, masses and scale μ in order to have dimensionless variables

$$\begin{aligned} k^2 &\to K^2 \equiv \left(\frac{k}{2\pi T}\right)^2 \\ m^2 &\to M^2 \equiv \left(\frac{m}{2\pi T}\right)^2 \\ \mu^2 &\to \Omega^2 \equiv \left(\frac{\mu}{2\pi T}\right)^2 \end{aligned} \tag{2}$$

we arrive to

$$J(M;a,b) = T(2\pi T)^{3+2a-2b} 2\Omega^{2\varepsilon} \sum_{n=0}^{\infty} \int \frac{\mathrm{d}^D K}{(2\pi)^D} \frac{(K^2)^a}{[K^2+(n+1)^2+M^2]^b} \tag{3}$$

where the sum was also arranged.

Using the known formulae for dimensionally regularized integrals [3] we find that

$$J(M;a,b) = T(2\pi T)^{3+2a-2b}\, 2\Omega^{2\varepsilon} \frac{\pi^{D/2}}{(2\pi)^D} \frac{\Gamma(\frac{D}{2}+a)}{\Gamma(\frac{D}{2})} \frac{\Gamma(l)}{\Gamma(b)} S(M,l) \tag{4}$$

where

$$S(M,l) = \sum_{n=1}^{\infty} \frac{1}{[n^2+M^2]^l} \tag{5}$$

and $l = b-a-D/2$. By taking the temperature to be larger than all the masses and energies in the problem, that is, by assuming that $M^2 < 1$, we can use the binomial expansion in order to evaluate $S(M,l)$. Therefore, $J(M;a,b)$ can be written as

$$\begin{aligned} J(M;a,b) &= T(2\pi T)^{3+2a-2b} 2\Omega^{2\varepsilon} \frac{\pi^{D/2}}{(2\pi)^D} \frac{\Gamma(\frac{D}{2}+a)}{\Gamma(\frac{D}{2})\Gamma(b)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} M^{2n} \zeta(2l+2n)\Gamma(l+n) \\ &= T(2\pi T)^{3+2a-2b} \frac{1}{4\pi^{3/2}} \left(\frac{\mu^2}{\pi T^2}\right)^{\varepsilon} \frac{\Gamma(\frac{3}{2}-\varepsilon+a)}{\Gamma(\frac{3}{2}-\varepsilon)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} M^{2n} \\ &\quad \times \zeta(2b-2a-3+2\varepsilon+2n)\Gamma(b-a-\tfrac{3}{2}+\varepsilon+n). \end{aligned} \tag{6}$$

As is well known, the divergencies present in the Feynman integrals will show up in dimensional regularization as poles when $\varepsilon \to 0$. In integrals of the form (1), the existence of divergencies is tightly related to the relation between the powers a and b, as one can see for example by power counting. The gamma function $\Gamma(z)$ is an analytic function of z with simple poles at the points $z = -m$ (for $m = 0, 1, 2, \ldots$). But it is clear that the condition $b - a - \frac{3}{2} + n = -m$ is impossible to satisfy because a, b and n are integers. Also, the Riemann's zeta function $\zeta(z)$ has a simple pole at $z = 1$. This implies that for $n = 2 + a - b$ the zeta function has a singularity. Since $n \geqslant 0$, this equality only holds if $b - a \leqslant 2$. In this case, we have that to order ε

$$\begin{aligned} \zeta(2b-2a-3+2\varepsilon+2n) &= \zeta(1+2\varepsilon) \\ &= \gamma_E + \frac{1}{2\varepsilon} + \mathcal{O}(\varepsilon) \end{aligned} \tag{7}$$

where γ_E is the Euler's constant. In the same way, expanding the gamma functions appearing in equation (6) to order ε

$$\begin{aligned} \Gamma(z+\varepsilon) &= \int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-t} t^{-z} t^{\varepsilon} \\ &= \int_0^{\infty} \mathrm{d}t \mathrm{e}^{-t} t^{-z} + \varepsilon \int_0^{\infty} \mathrm{d}t \mathrm{e}^{-t} t^{-z} \ln t \\ &= \Gamma(z)\{1+\varepsilon\psi(z)\} \end{aligned} \tag{8}$$

and using the property of the ψ function [4]

$$\psi(A+a) = \psi(A) + \sum_{p=0}^{a-1} \frac{1}{A+p} \tag{9}$$

we find that

$$J(M;a,b) = T(2\pi T)^{3+2a-2b}\frac{\Gamma(\frac{3}{2}+a)}{2\pi^2(b-1)!}\Bigg\{(-1)^{a-b}\frac{\sqrt{\pi}M^{4+2a-2b}}{2(2+a-b)!}$$
$$\times\left[\frac{1}{\varepsilon}+\gamma_E-\ln\left(\frac{4\pi T^2}{\mu^2}\right)-\sum_{p=0}^{a-1}\frac{2}{3+2p}\right]$$
$$+\sum_{\substack{n=0\\ n\neq a-b+2}}^{\infty}\frac{(-1)^n}{n!}M^{2n}\zeta(2b-2a-3+2n)\Gamma\left(b-a-\frac{3}{2}+n\right)\Bigg\} \qquad (10)$$

for $b-a\leqslant 2$.

For $b-a>2$, the argument of the gamma function, $b-a-\frac{3}{2}+n$, is always larger than 1. Therefore, $\zeta(z)$ and $\Gamma(z)$ are not singular when $\varepsilon\to 0$ and thus we can simply set $\varepsilon=0$ in equation (6). Therefore,

$$J(M;a,b) = T(2\pi T)^{3+2a-2b}\frac{\Gamma(\frac{3}{2}+a)}{2\pi^2(b-1)!}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}M^{2n}\zeta(2b-2a-3+2n)$$
$$\times\Gamma(b-a-\tfrac{3}{2}+n). \qquad (11)$$

These expressions are well suited for algebraic programming. In fact, it is easy to write a small program in for example MAPLE V [5] in order to evaluate equations (10) and (11). In this way one is able to evaluate the integrals to the desired order in M^2.

As a simple example, let us consider the integral (see [2], equation (3.11))

$$I(m) = T\mu^{2\varepsilon}\sum_{n=-\infty}^{\infty}\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{[k^2+\omega_n^2+m^2]}$$
$$= T\mu^{2\varepsilon}\int\frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{[k^2+m^2]}+J(m;0,1) \qquad (12)$$

where we have used the notation of equation (1). The first integral is finite and equal to $-(1/4\pi)mT$. And since in this case $b-a\leqslant 2$ we can apply equation (10), obtaining

$$I(m) = -\frac{mT}{4\pi}+\frac{T^2}{\pi}\Gamma\left(\frac{3}{2}\right)-\frac{\sqrt{\pi}}{2}M^2\left[\frac{1}{\varepsilon}+\gamma_E-\ln\left(\frac{4\pi T^2}{\mu^2}\right)\right]$$
$$+\zeta(-1)\Gamma\left(-\frac{1}{2}\right)+\frac{M^4}{2}\zeta(3)\Gamma\left(\frac{3}{2}\right)$$
$$= -\frac{mT}{4\pi}+\frac{T^2}{12}-\frac{m^2}{16\pi^2}\left[\frac{1}{\varepsilon}+\gamma_E-\ln\left(\frac{4\pi T^2}{\mu^2}\right)\right]+\frac{m^4\zeta(3)}{8(2\pi)^4T^2} \qquad (13)$$

where equation (2) has been used. This result coincides, up to order ε, with that of equation (3.12) in [2]. Here we have also included the order m^4 term. The temperature-dependent part of the one-loop effective potential can be obtained from here by integrating in m equation (13) times m.

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